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Developing equitable elementary mathematics classrooms through teachers learning about children's mathematical thinking: Cognitively Guided Instruction as an inclusive pedagogy

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HIGHLIGHTS

- Teachers developed a clearer insight into children's mathematical understanding.
- Teachers recognised the importance of this knowledge.
- Previous perceptions of children's abilities were challenged.
- Teachers became more aware of their capacity to support all learners.
- Teachers recognised this learning as an ongoing process.

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ABSTRACT

This paper reports on a study carried out in Scotland which involved introducing the principles of Cognitively Guided Instruction (CGI) to 21 mainstream elementary teachers. It considers the effects of developing CGI in classrooms focussing on teacher learning and particularly their capacity to support all learners. The findings demonstrate teachers' awareness of their own learning and how increased understanding of children's mathematical thinking left them better placed to support all learners. The study highlights the importance of developing teachers' knowledge of children's mathematical thinking in order to promote inclusive practices with CGI providing a useful framework for this professional development.

1. Introduction

Equitable practice in mathematics teaching acknowledges the involvement of all students in making sense of their mathematical learning within classroom communities that are respectful of difference (NCTM, 2000). This position reflects international moves exemplified by the Salamanca Statement (UNESCO, 1994) and driven by legislation that seeks to advance social justice, equity and inclusion. This agenda has been progressed in the United States through No Child Left Behind (2001) and the Individuals with Disabilities Education Act (2004); in the UK, in England through the Special Educational Needs and Disability Act (2001) and the English Code of Practice (DfES, 2001); in Scotland through the Standards in Scotland's Schools etc. Act (2000), the Additional Support for Learning Act (2004 as amended 2009) and Supporting Children's Learning: Code of Practice (Scottish Government, 2010).

International studies on inclusive education have shown a continuum of educational provision, at a structural level, intended to accommodate all learners through appropriate allocation within that continuum (Armstrong, Armstrong, & Spandagou, 2010; Muskens, 2011; Rix, Sheehy, Fletcher-Campbell, Crisp, & Harper, 2013). A more radical view of inclusion recognises inclusive education as the restructuring of schools so that they become places for all children (Allan, 2010; Slee, 2011). If we are to have schools for all children then we must have classrooms in which everyone is a member of a 'community of learners' (Thomas, 2013). This requires a pedagogical approach intended for everyone. Traditional approaches to meeting the challenge of diversity in classrooms suggest that teachers need to access a specialist knowledge base or even a specialist pedagogy (Florian, 2009; Porter, 2005). An

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alternative view, argued for by proponents of inclusion, suggests that there may be a commonality to effective teaching practice that is of benefit to all learners (Norwich & Nash, 2011). The development of inclusive practice in relation to pedagogy becomes crucial if one considers the classroom as a place for everyone (Hart, Dixon, Drummond, & McIntyre, 2004). This is in contrast with the identification of some learners as requiring something additional and different frequently beyond the classroom and sometimes beyond the regular school (Florian & Black-Hawkins, 2011).

The application of the concept of inclusive pedagogy to the teaching of mathematics in the elementary classroom reflects a principled approach to teaching in a specific domain requiring knowledgeable teachers responsive to the needs of all students (Greer & Meyen, 2009; Jordan, Schwartz, & McGhie-Richmond, 2009). Responding to the needs of individuals on the basis of teachers' knowledge of children's thinking is challenging and complex and is connected to the type of professional development that teachers undertake (Jacobs, Lamb, & Philipp, 2010). Knowledge of children's mathematical understanding is a powerful instructional pointer (Fennema, Franke, Carpenter, & Carey, 1993) which facilitates an educational response to the learning needs of all pupils (Behrend, 2003; Empson, 2003). Cognitively Guided Instruction (CGI) provides a research-based framework, developed at the University of Wisconsin-Madison (Carpenter, Fennema, Franke, Levi, & Empson, 1999), for teachers to learn about children's mathematical thinking. This article reports on the introduction of the principles of CGI to mainstream classrooms in the UK. It focuses on the learning of twenty-one teachers in Scottish primary (elementary) schools. It considers what they gained from this professional development and specifically the extent to which they felt better equipped to support the learning of all children.

1.1. Inclusive pedagogy

A traditional response to support children who struggle in their learning follows a medical model in which the problem is viewed as a deficit within the child to be remediated. Such reductionist approaches are fundamentally rooted in behaviourist, lock-step approaches to teaching and assessment that historically have been a feature of special education (Dyson, 2001; Goddard, 1997; Thomas & Loxley, 2007). Rather than permitting the purpose of identification to be separatist (Tomlinson, 1982) and responding to this identification and assessment on the basis of individualistic interventions (Dyson, 2001), teachers and managers within schools might consider how they conceptualise learning difficulties not solely in terms of the needs of the individual but also from a pedagogical perspective. Ainscow (1994) has argued that the individualisation which traditionally lies at the heart of many interventions designed to respond to children's needs is problematic because it encourages a focus on the individual rather than on the curriculum, thus failing to recognise issues of instruction as potentially problematic. McIntyre's paper (2009), published posthumously, contains an editor's note that merits reproduction, in which an inclusive pedagogy is defined as:

> a collaborative approach to teaching based on the idea that all children can learn together, and that participation in learning requires responses to individual differences among learners that do not depend on ability labelling or grouping, or the withdrawal of the learner for additional classroom support (p. 603).

This position questions the usefulness of distinguishing between groups in order to classify, instead drawing attention to the need to consider classroom conditions and contexts that facilitate effective learning for all pupils (Ainscow, 1999). Within this perspective instructional decisions are not made on the basis of categorical differences but rather are based on detailed knowledge of the conceptualisations of individual children and the degrees of support required (Empson, 2003; Stough & Palmer, 2003). How this detailed knowledge then informs teaching and how teaching is structured to support all learners become important pedagogical decisions. Recognising the development of inclusion as linked to the development of pedagogy is key. In the US a case has been made for focussing on the improvement of teaching through focussing on children's learning (Stigler & Hiebert, 2009). This argument connects with Japanese models of teacher development that focus on interpreting pupils' learning rather than on techniques of teaching (Watanabe, 2002). This practice has been developed in Europe and in the UK (Dudley, 2012; Norwich & Jones, 2014) and specifically around children with learning difficulties as a way of developing more inclusive practice (Ylonen & Norwich, 2012).

A pedagogy in which the 'transformability' of every learner is recognised supports the learning capacity of every individual and the development of an inclusive culture (Hart et al., 2004). Florian and Black-Hawkins (2011, p. 2) describe this as requiring a 'shift in pedagogical thinking' away from what works for most learners along with something 'additional or different' for some learners towards creating opportunities in which all learners are able to participate. Inclusive pedagogy rests in a complex interplay involving teachers' knowledge and beliefs about: individual learners, teaching, self-efficacy and the pedagogical decisions and action which ensue (Jordan et al., 2009; Lalvani, 2013). Such practice also requires domain specific knowledge without which teachers may be ill-equipped to support all learners (Ball, Thames, & Phelps, 2008; Hiebert, Gallimore, & Stigler, 2002; Ma, 1999).

1.2. Pedagogical knowledge and beliefs

Pedagogy goes beyond the act of teaching and includes 'the ideas, values and beliefs by which that act is informed, sustained and justified' (Alexander, 2008, p. 4). Recognising pedagogy as teaching acts influenced by values and beliefs helps to distinguish an inclusive pedagogy from inclusive practices, the latter potentially being seen to address issues of equity through responses to legislation and procedural imperatives (Dyson, 2001). Implicitly an inclusive pedagogy recognises teachers' attitudes and beliefs as key elements of an inclusive approach.

The success of mathematics education initiatives is dependent on encouraging teachers to make changes in their beliefs (Lloyd, 2002, p. 150). Initiatives that seek to develop mathematical teaching which can be viewed as part of a reform movement in mathematics instruction (Fuson et al., 2000) are, to a considerable extent, dependent on the identification of effective strategies for professional development at every level within a school (Carpenter et al., 2004). Such initiatives prospectively facilitate significant shifts in teachers' beliefs (Lloyd, 2002). When teachers engage with innovative, or at least unfamiliar, practices there is potential for personal as well as professional development; opportunities arise in which existing pedagogical beliefs are challenged and questioned (Janssen, Westbroek, & van Driel, 2014; Makinen, 2013; Waitoller & Kozleski, 2103).

Although it has been acknowledged that no teacher alone has the expertise to meets the needs of every learner (Garderen, Scheuermann, Jackson, & Hampton, 2009) the notion that there is a unique body of pedagogical knowledge required by teachers to support particular learners has been challenged (Fletcher-Campbell, 2005; Jordan et al., 2009; Lewis & Norwich, 2001). This argument maintains that the interpretation of children's understanding is a crucial element in developing inclusive practices and recognises the application of knowledge of children's

conceptualisations as more useful than the identification of learner deficits in themselves. This view has been represented in the domain of literacy (Elliot & Gibbs, 2008).

In mathematics education the proposition that knowledge of children's mathematical thinking should inform instruction is well-established internationally. In the US this is made explicit in the Common Core State Standards and exemplified by pedagogical practices which recognise this (Boaler & Humphries, 2005; Carpenter et al., 1999; Fennema & Romberg, 1999; Jacobs et al., 2010; Yackel & Cobb, 1996). Similarly, the Maths Recovery programme developed in Australia and introduced into the UK, Ireland and Canada (Wright, Martland, & Stafford, 2006), work developed in the Netherlands through the Freudenthal Institute (Gravemeijer, 1997) and connected work in the US (Fosnot & Dolk, 2001) coalesce around the principle of mathematics learning as a sense-making process with teachers' recognition of children's mathematical understanding as crucial. This is an important element of teachers' pedagogical content knowledge (PCK) as posited by Shulman (1986). PCK is recognised as a complex construct and in relation to mathematics teaching the mathematical component is a crucial one (Ball et al., 2008; Depaepe, Verschaffel, & Kelchtermans, 2013). In a study which sought to understand effective teachers in numeracy the researchers set out a model which reflects the interplay between teachers' knowledge and beliefs, classroom practices and pupils' responses (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997, p. 18). This framework was characterised by teachers' subject knowledge, their knowledge of teaching approaches and their knowledge of learners, and how these come together to inform instruction. Askew et al.'s framework is situated within a sociocultural paradigm and allows for a consideration of both teacher and pupil development as a participatory process situated within a 'community of learners' (Lave & Wenger, 1991). The application of this model for teacher learning is considered relevant to the development of both more effective mathematics teaching (Lewis, Perry, & Hurd, 2009) and more inclusive classrooms (Ylonen & Norwich, 2011).

1.3. Teacher learning as situated activity

Learning with understanding is often conceptualised on the basis of the knowledge of the individual (Carpenter & Lehrer, 1999); however there is a growing view that it is useful to consider learning with understanding as an emerging process that functions within a community of learning (Carpenter et al., 2004; Rogoff, Matusov, & White, 1996; Yackel & Cobb, 1996). Within such a community, learning is viewed as a generative and situated process (Lave & Wenger, 1991) which has implications beyond the individual child and towards the development, both personal and professional, of all within that community.

Research evidence about teachers' learning about mathematics teaching shows effective learning is situated in classroom interactions (Boaler, 2002; Empson & Junk, 2004; Fennema et al., 1996; Hiebert et al., 1997; Lampert, 2001) and that teachers learn about supporting pupils who struggle in their learning through purposeful interactions (Behrend, 2003; Watson, 1996, 2001). Mathematics teaching that is informed by knowledge derived from research can lead to improved practice (Empson & Junk, 2004; Fennema et al., 1993; Franke & Kazemi, 2001). However there is a concern that this knowledge base is not seen as relevant to the learning of all pupils with a view that some additional or different pedagogical knowledge is required for some learners as opposed to recognising what is common and available to everyone (Florian, 2009; Ylonen & Norwich, 2012). The development of this aspect of PCK has the potential to inform classroom practice (Askew et al., 1997). It is an interactionist response which recognises the importance of learning and development from the point of view of the teacher as well as that of the child. In practice this might mean that concerns about meeting a child's needs through a process of assessment, perhaps driven by determining what a child is unable to do, becomes displaced by a more dynamic view of assessment that uses information about a child's conceptualisations to inform instruction (Jacobs et al., 2010; Stringer, 2009; Watson, 1996). A pedagogy that sustains assessment as a dynamic process illuminates learners' needs on the basis of actual current knowledge and understanding, rather than on the basis of identifying any gap between where a child is, or should be, within a particular curricular framework.

2. Cognitively Guided Instruction

Cognitively Guided Instruction (CGI) (Carpenter et al., 1999) was the professional development programme used within the project. CGI is built on the thesis that children come to school with a great deal of intuitive and informal mathematical knowledge which serves as the basis for developing more formal understanding (Carpenter et al., 1999). It is not a prescriptive pedagogy or an acquirable teaching technique. It is a principled approach to teaching mathematics which recognises mathematics learning as a sense-making activity. In practice CGI involves the use of arithmetical word problems. Teachers are provided with two related research-based frameworks: word problem types and children's solution strategies. As pupils engage with particular problems teachers learn to interpret their solution strategies and use this analysis to inform their teaching. In this way teaching follows constructivist principles and is based on building on the sense that children are making of problems; teachers focus on what students know and understand and help them to build on that understanding.

Focussing on children's understanding provides a context for teachers to develop their own pedagogical knowledge. Thus teacher learning becomes a dynamic process situated within classroom interactions and interpretations. Increasing teachers' knowledge of students' thinking helps them to design better instructional tasks and to support student learning more effectively (Steinberg, Empson, & Carpenter, 2004). CGI provides a framework for developing this understanding. It is not a method as such and there is no single way in which it comes to be applied in practice, however there are common features to CGI classrooms (Carpenter et al., 2004). These classrooms reflect a sociocultural perspective with pupil and teacher learning situated within a process of dynamic activity (Rogoff et al., 1996).

Research and professional development work involving CGI have been almost exclusively American. There has been some work with CGI in Iceland (Steinthorsdottir & Sriraman, 2009) and there is ongoing work with pre-service teachers in Israel (Steinberg, 2013). To date CGI has not been developed or researched within mainstream classrooms in the UK. There have been very few studies involving CGI with pupils with learning difficulties, those carried out have been positive. These studies, in the UK (Moscardini, 2010) and in the USA (Behrend, 1994, 2003; Empson, 2003) found that pupils with learning difficulties employed the same intuitive strategies, outlined by the original researchers, as typical children. There is evidence of the positive impact that CGI has in developing teachers' knowledge, specifically in terms of pedagogy and knowledge of students' mathematical thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Empson & Junk, 2004; Peterson, Fennema, Carpenter, & Loef, 1989). CGI represents a situationally-based form of professional development, a feature of which is the latitude that teachers have to develop their own learning and understanding of children's mathematical thinking. This openness is particularly relevant given current curricular developments in Scotland.

3. The Scottish context

The study was set within the context of important developments within the Scottish educational system. A new curriculum, 'Curriculum for Excellence' (CfE) was recently introduced in Scotland (Scottish Government, 2009). In principle this is an inclusive curriculum that is responsive to worldwide calls for inclusive practices in education and specifically to recent Scottish legislative demands. A feature of CfE is its aim to develop critical thinking, communication and autonomy. The extent to which all children, including those who struggle in their learning, might be afforded the opportunity and autonomy to fulfil these aims may be linked to teachers' knowledge and beliefs and remains largely unexplored as far as mathematics teaching in Scottish primary classrooms is concerned. Curriculum for Excellence does not explicitly refer to inclusion but it is clear that this is a curriculum for all children (Allan, 2008).

4. Study aims

The aim of the study was to develop teachers' understanding of children's mathematical thinking through an introduction to Cognitively Guided instruction and to explore if and how they used this knowledge to support all learners.

5. Study design

The study was a qualitative one designed over three phases to support a comparison of pre- and post-intervention measures. CGI was introduced as a vehicle for learning about children's mathematical thinking, and not as a course to be subsequently evaluated. Data were gathered during each phase of the study.

Phase 1

This phase aimed to determine teachers' existing knowledge base and accounts of their current practice prior to the introduction of CGI. It involved individual semi-structured interviews and an analysis of teachers' current planning and assessment records.

Phase 2

In this phase teachers undertook two days professional development in CGI. This focussed on developing an understanding of two frameworks: word problem types for addition and subtraction and children's solution strategies (Carpenter et al., 1999). The development days occurred 7–10 days apart so that teachers could apply their learning in practice and feedback. Teachers then had a 12 week period of implementation during which time they carried out CGI informed lessons at least once a week. CGI is not a prescriptive pedagogy and teachers were encouraged to develop it in their classrooms as they saw fit. They were not asked to focus on specific children, whether and how they identified and responded to particular children was left open. Classroom observations took place towards the end of the implementation period. Continued support was available during this period through online discussion and discussion following classroom observations.

Phase 3

This was the final phase. It involved semi-structured interviews and an analysis of classroom-generated artefacts of pupils' problem-solving solutions and engagement and teachers' records of CGI sessions. The interviews included opportunities for teachers to discuss children's solution strategies and to refer to examples of children's work.

5.1. Ethical procedure

The study conformed to the requirements of the University of Strathclyde Ethics Committee. Consent for participation was approved by Local Authorities. Participating teachers were provided with information sheets outlining the details of research study and the nature of their participation. Each participant completed and returned an individual consent form.

5.2. Sample group

The participants were twenty-one mainstream primary teachers from ten primary schools within two neighbouring Scottish Local Authorities. The Local Authorities nominated the participating schools. The sampling strategy was random at the point of schools self-selecting participants. It was purposeful (Patton, 2002) in that teachers had to be in a position to implement CGI sessions on a regular basis. The mean length of service of participants was 14.1 years, the longest service being 34 years and the shortest 3 years. None of the sample group had any prior experience of CGI. The majority of teachers had undertaken no professional development in numeracy in recent years. One Primary 7 (11 year olds) teacher withdrew from the study after the first professional development session. The reason given was that he felt that CGI would 'slow down the pace of learning' and that it was 'not relevant' for his class.

5.3. Content of professional development sessions

Teachers underwent two days professional development in CGI. The content of the professional development sessions was drawn from activities and materials set out in the CGI Workshop Leaders manual. The sessions adhered to the recommended sequence by introducing frameworks of addition and subtraction problem types and children's solution strategies related to these problem types. These frameworks form a basis for understanding children's mathematical thinking. Due to the limited time available professional development was restricted to addition and subtraction. It was deemed more useful to cover content in depth rather than adopting a wide and shallow approach.

5.4. Data collection

Data were gathered from a range of sources.
In Phase one:

- 21 teachers were interviewed prior to professional development in CGI. Interviews were taped and transcribed.
- Teachers provided current written lesson planning and assessment data for their current classes.

The interviews explored teachers' perceptions of their own mathematical knowledge, their knowledge of children's solution strategies and their knowledge, beliefs and practices about teaching numeracy. The lesson plans and assessments provided information on practice and the extent to which practice, prior to development in CGI, was informed by knowledge of children's understanding.

In Phase two teachers developed CGI in their classrooms. The following data were gathered:

- teachers' notes and recordings of classroom episodes; these included fieldnotes, photographs, videoclips and examples of student work which constituted a dynamic assessment of

pupils' mathematical activity. Teachers were provided with a framework for recording CGI sessions. This involved recording problem types used and children's solution strategies.
- notes of teachers' discussions, email and telephone correspondence with individual teachers
- researcher's fieldnotes and journal
- observations by the researcher of CGI sessions classroom in every classroom.

In Phase three teachers were interviewed again after having implemented CGI in their classrooms. These interviews explored teachers' learning following the intervention and also provided the teachers with an opportunity to discuss student-generated artefacts.

5.5. Analysis

Data were analysed adhering to an iterative method, 'Framework', developed at the National Centre for Social Research (UK). Framework is a matrix-based analytic method that permits a rigorous and systematic analysis of data. At each stage of the analysis it is possible to work at increasing levels of abstraction with the original data being accessible at each stage of this process (Ritchie, Spencer, & O'Connor, 2003).

Interviews were transcribed, then read and re-read. Topics were identified and a coding system developed. The final categories that emerged were: Knowledge and Beliefs – which reflected teachers' subject knowledge, pedagogical knowledge, knowledge of learners and beliefs about learning and teaching of numeracy. Professional Development-related to issues of professional development and teachers' self-efficacy; Elements of Instruction-related to observable classroom practices and procedures, pedagogy and didactics. Once the transcripts were coded, thematic charts were developed following the framework outlined by Ritchie and Lewis (2003). Data were decontextualized and then recontextualized (Tesch, 1990) within these charts so that similar content could be located together. The thematic charts supported the analysis of data across categories by participant but importantly they also facilitated a cross-sectional analysis of each category. To ensure reliability of the coding random-sampled transcripts were cross-checked by blind-coding. The final interviews were coded using the same categories. These data were ordered within the initial interview thematic charts permitting a pre- and post-intervention comparison.

Hardcopy and observational data were analysed alongside interview data in the thematic charts to develop a descriptive analysis of the underlying process (Ritchie et al., 2003). Hardcopy and observational data comprised: class planning and pupil assessment records; hardcopy and photographic evidence of pupils' work; videoclips of classroom episodes; teachers' fieldnotes; fieldnotes of researcher-observed CGI sessions; researcher journal comments; email correspondence. The fieldnotes of the researcher-observed CGI sessions were an important part of the analytical process, these observations served to validate or contest teachers' recordings and interpretations of the CGI sessions. Gathering evidence from a range of sources permitted data to be analysed through a wide lens and also allowed each element to be analysed individually. These multiple perspectives ensured that a rich picture emerged (Ritchie, 2003).

5.6. Limitations

The CGI development sessions took place within particular time constraints. In the US the allocated time for introducing CGI is 40 h. Study participants received the equivalent of two in-service days which meant that participants had limited opportunities to discuss and reflect on practice collaboratively.

The open questions within the semi-structured interviews were designed to allow teachers to articulate how they viewed themselves as teachers of mathematics and to reveal their knowledge and beliefs about children's mathematical learning. In order to minimise a bias towards discussion of children who struggled in their learning the teachers were not questioned about specific groups of learners. Questions were about mathematics teaching and learning in general. Any comments about particular children were expressed by the teachers within the context of these open questions; probes were used only when teachers did not discuss issues relating to diversity in their classroom.

The study recognised the complex interplay that exists between pupil engagement, teachers' knowledge and beliefs and current classroom practices and procedures. The relationships between these elements are understood as being socially and culturally situated. In this respect it is not possible to extrapolate the findings relating to one particular group to similar groups within different contexts. Any conclusions drawn that relate to the sample group are limited to an expression of potential for any similar group within a different setting.

6. Findings

The central aim of the study was to determine teachers' learning following professional development in CGI and the extent to which, if at all, it supported the development of inclusive pedagogy. The study produced a rich and varied data set. Examples from the thematic analysis drawn from teachers' notes, pupils' work, classroom observations and interviews have been used to evidence changes in teachers' thinking. Consideration has also been given to evidence of factors that might constrain this development. The findings are presented thematically under the following headings: Teachers' pedagogical positioning – accounts of classroom practice; Teachers' accounts of noticing children's strategies; Using knowledge of children's strategies to inform teaching; Teachers' reflections on interactions; Evidence of moving towards an inclusive pedagogy. All names have been anonymised. Classes have been identified by stage, the age range of pupils at the start of the school year is provided as a frame of reference.

6.1. Teachers' pedagogical positioning – accounts of classroom practice

Although in the initial interviews teachers described themselves as 'facilitators' most teachers' accounts were of demonstration. Prior to professional development in CGI teachers described their practice in terms of knowledge transmission and showing children how to solve problems. Although the study was not focussed towards struggling learners it was notable that a sense of transmission teaching was particularly evident in teachers' accounts of supporting children who required support. In the initial interviews several teachers described an organisational response for supporting struggling learners which usually involved additional adult support and/or separation from rest of the group, for example:

> I have a classroom assistant and … and (she) tends to work with children individually. (Julie, Primary 5/6, 8–10 year olds)

Examples of support through consideration of teaching included different ways of explaining which often involved repetition of content, for example:

> I try to explain it in different ways if they didn't get it the first time. (Karly, Primary 4, 7–8 year olds).

Following the development of CGI in their classrooms, every teacher described changes in classroom practice that demonstrated increased opportunities for all children to share their understandings and to lead in their learning as opposed to being the passive recipients of knowledge.

> I never thought of sitting round in a group and sharing ideas. (Mina, Primary 2, 5–6 year olds)

Interview data were supported by classroom observations by the researcher and by teachers' own records which showed that all children were taking part in the activities with teachers being surprised at the strategies used by particular children. There was evidence that some teachers were beginning to view the engagement of particular pupils from a different perspective. A Depute Headteacher commented:

> What I found actually the most important was that what I expected from the children is not what I got. In every class there are children who really shone that I wouldn't have expected.

Prior to CGI development only one teacher of a class of 5–6 year olds described the free use of concrete materials. None of the teachers in the upper stages of any of the schools described materials being used autonomously by pupils. Following the development teachers described materials being used more flexibly, this was supported by classroom observations. In the initial interviews problem-solving was generally described as being taught discretely, sometimes on specific days, often described as a weekly activity and generally not associated with numeracy. Arithmetical calculation was accounted for more through explicit instruction rather than through problem-based activity. 'Active learning' in mathematics was frequently encouraged but sometimes focussed on kinaesthetic rather than cognitive activity. One teacher gave an example of active learning as 'children doing jumping jacks to the times tables' (Roseanne, Primary 4/5, 7–9 year olds). Prior teaching was beginning to be seen as a barrier to learning by several teachers, one commented:

> I definitely learned more … and it really made me think that a lot of their problems, this sounds dreadful, have been caused by the way we teach. (Julie, Primary 5/6, 8–10 year olds)

6.2. Teachers' accounts of noticing children's strategies

Every teacher kept detailed accounts of their CGI sessions. These included records of the problems given and analyses of children's solution strategies. Most teachers went beyond this by recording their reflections on the activities. The format of these records varied from teacher to teacher, most kept journal-type records along with examples of pupils' work. Two teachers video-recorded pupils working on problems. In interview one teacher described using her recordings for sharing her learning with colleagues in her school.

The quality of teachers' observations and accounts of children's solution strategies was closely connected to their knowledge of what to look for. In the initial interviews teachers' knowledge of children's solution strategies was limited. Nineteen of the twenty-one teachers found it difficult to describe how children might solve a simple addition problem, for example $3 + 6$. Two early stage teachers gave some account of what children might do by describing a count-all strategy, with the exception of these two accounts no teacher was able to elaborate how pupils might use their fingers or materials. No teacher described actively looking for what it was that children were doing with their fingers or materials to solve problems; there was no evidence of teachers using knowledge of children's strategies to inform teaching. Following the application of CGI in the classroom teachers were surprised by what they were learning about pupils. A senior manager with over 30 years teaching experience said:

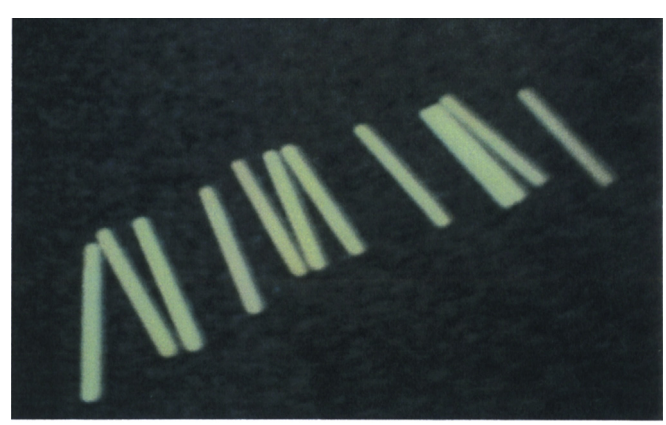
> I've learned that I didn't really focus on the way they were thinking. I knew what I was teaching and I was very confident that I could teach children but when I actually used CGI the children were not working the way I expected them to … I think it is probably the quickest way of finding out exactly where each child is at.

Discussion with teachers about their observations of children's solution strategies revealed a deeper insight into children's understanding. Referring to a photograph and her annotated notes (Fig. 1) one teacher described being surprised at a pupil's intuitive strategy particularly because she considered him to have learning difficulties. The problem was – In the fruit shop melons are stacked in layers in a crate. There are twelve melons on each layer. How many melons are in three layers? She explained that she watched him model the problem and skip count in fours, something which she did not know he was able to do. This observation gave her valuable information about the pupil's understanding. It also provided an opportunity to share and discuss with colleagues how she gained this insight. Teachers' noticing of children's solution strategies was a significant development in their understanding:

> … the things they were coming up with, their different ways of doing it opened my eyes … on my planner I would have ticked the box that they have completed this, yet Matthew [hardcopy examples presented] could barely explain. Robert was much further on as he could count. Before I would have said that they could both add. I would never have expected there to be as wide a gap. (Shelley, Primary 3, 6–7 year olds).

6.3. Using knowledge of children's strategies to inform teaching

Prior to professional development in CGI there was evidence from the initial interviews, supported by the planning and assessment documents, that every teacher knew where individual



This example was really interesting . This pupil has learning difficulties and I thought he had it wrong but when he explained his reasoning he was right. Three of the sticks represented the layers. He only had three sticks in between each layer but said that each sticks represented 4 melons. He then added 4+4+4+4……. Until he reached 36.

Fig. 1. Scanned copy of original annotated example brought in by teacher for discussion.

children were in terms of planning frameworks and external curricular material. However there was little evidence to show that teachers were using their knowledge of children's conceptual understanding to inform their teaching. In the initial interviews only two of the participants were able to give detailed accounts of what particular children were doing when solving arithmetical problems. There was some evidence that there was an issue of teachers lacking a conceptual basis for describing children's mathematical thinking. The majority of teachers described determining the next steps in children's learning by following external planning frameworks rather than basing their instructional decisions on their knowledge of children's thinking. Following professional development in CGI teachers were beginning to describe a more dynamic view of assessment:

> I now have a better understanding of how children think. I am no psychologist but I have a better handle on how the children are actually working out the problems. (Carol, Primary 6, 9–10 year olds)

The detail of the teachers' accounts post-professional development was significant. Teachers described noticing how children were counting, if they were touching each item, nodding at items as they counted, skip counting, modelling using materials. Some teachers were beginning to use this knowledge to inform their teaching. One teacher recorded detail in her notes that was not evident in the interviews or in planning or assessment documents before the professional development:

> Kevin is able to count on and count back with concrete materials but he doesn't count in tens. He should be given more problems that encourage him to count in tens. (Karly, Primary 4, 7–8 year olds).

Teachers were beginning to use pupils' solutions as a way of understanding their mathematical thinking and used this insight to inform teaching. One teacher, Lesley, recorded in her notes:

> I have been surprised and impressed at the way in which the children solve or attempt to solve problems. When Callum, (a 6 year old pupil in an educational support class), was presented with the problem: Mrs Fraser has 6 apples. She eats 3 at lunch. How many apples does she have left? He drew a bin, 3 apple cores and then 3 more full apples to make 6 and got his answer of 3 that way. Had I not been using CGI I would probably have been shouting '6 takeaway 3' rather than allowing him to work the problem out in his own, very unique way.

Lesley had previously noted that Callum struggled with missing addend problems. Her observation showed that he could use inverse operations and she decided to see how he would deal with a challenging missing addend problem. Fig. 2 shows how Callum solved this problem ($4 + x = 7$) expressed as 'Callum has four pairs of glasses. How many more pairs does he need to have seven pairs of glasses?' It was a contextually relevant problem for Callum who frequently broke his glasses. In his drawing, without instruction, Callum drew out the story by drawing four pairs of glasses and then his mum going to the shops to buy more pairs, recording these one at a time and counting on to seven. All the elements of his solution are visible in the drawing. Lesley recorded Callum's solution strategy in her notes and used it as an example for discussion with colleagues on an inservice session.

Classroom observations showed that some teachers were not always picking up on what particular children were doing. For example in Kirsty's class a pupil solved a problem that involved

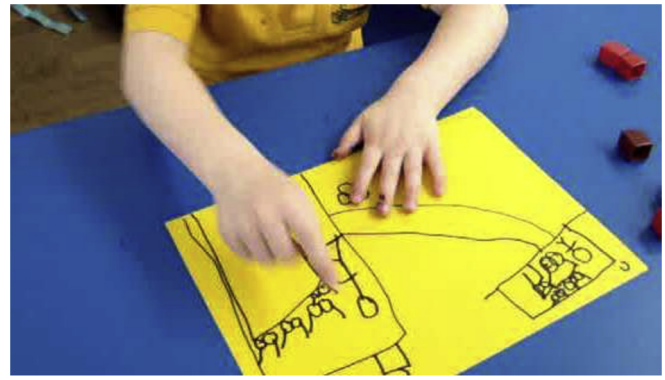


Fig. 2. Callum's account of how he solved a missing addend problem ($4 + x = 7$).

100–50 by drawing an array of 20 boxes (each box represented 5) and counting off in 5 s. She told the teacher she drew 100 boxes and took away 50, the teacher accepted this explanation even though the child's strategy was quite different and displayed more advanced mathematical understanding. In another class some children who had used number facts posters on the walls were deemed to 'know' the number fact without teacher probing. This aspect of teacher learning is recognised as requiring time and for some teachers it is a challenging step. Discussion with the teachers showed that at this early stage of the implementation some teachers were focussing more on whether they were 'doing it right' rather than trying to interpret what children were doing.

6.4. Teachers' reflections on interactions

In the final interviews teachers described how children were actively encouraged to discuss and share their thinking. Through listening to the children and watching what they were doing teachers were beginning to reflect on their practice and question their beliefs. Opportunities for teachers to reflect on what children were doing relates to the engagement of the pupils. In this respect the absence of particular comments is important. In other words, had pupils failed to engage, either at group or individual level, it is likely that teachers would have made this known to each other. It was notable, given the diversity in the classes in terms of age, ability and cultural differences, that not one teacher gave examples of pupils being reluctant or unable to participate in the CGI sessions. In particular, three of the classes involved children with English as an additional language. Two of which were in a school of 300 pupils of whom less than five had English as their first and only language.

Through observing and analysing children's mathematical behaviour as they engaged in CGI problems, some teachers commented that the children's performance did not always reflect their formal assessments. Several teachers emphasised the problem of pupils following taught procedures without understanding. This procedural competence without the commensurate conceptual understanding was viewed as increasingly problematic by teachers, particularly as children moved through primary school and came to be recorded as having achieved various attainment levels that did not necessarily reflect their conceptual understanding, as one teacher explained:

> it opened up a whole new way of thinking for me … it highlighted how little basic knowledge some children have, they are doing things they just don't understand (Andy, Primary 7, 10–11 year olds)

6.5. Evidence of moving towards an inclusive pedagogy

Prior to the introduction of CGI, twenty of the twenty-one teachers described themselves as confident and well-equipped to teach mathematics. This was on the basis of their subject knowledge, knowledge of resources or personal enjoyment of the subject. Some teachers felt more comfortable working with earlier stages. However more than half of these teachers described themselves as feeling less competent to support children who struggled in their mathematics learning, for example:

> therein lies my problem in ensuring that [children who struggle] actually understand (Anne, Primary 1/2, 4–6 year olds)

The final interviews revealed that by learning about children's mathematical understanding the teachers felt more equipped to support particular children in the context of the classroom rather than using this knowledge as a mechanism for their removal. Teachers' narratives showed that this learning was seen as situated classroom activity specifically within teacher–pupil interactions. It was notable that teachers recognised their own learning within this structure although there was less discussion of learning as a collaborative process amongst pupils. One teacher said:

> [I'm] much better placed to support all learners … I now have a better understanding of how children think. If I'm in Primary 7 next year and I have a child working towards (early level), then I'm definitely more equipped to support them. (Kirsten, Primary 7, 10–11 year olds)

There was evidence of a shift in some teachers' thinking about the structure of educational support. For example, one teacher began by working with a particular child on a one-to-one basis before deciding to include the CGI problems for this child within the context of classroom-based activity. Prior to CGI development some records of planning for individuals contained general and vague statements such as: 'keep practicing tables'; 'increase rate of work'; ' keep an eye on R & C'. These statements reflect a curriculum-led approach to teaching and intervention, a model which privileges the completion of work as set out in planners over the development of conceptual understanding. This is in contrast to the principle of teaching being planned on the basis of children's actual understanding, an argument which permeated the final interviews.

7. Discussion

The aim of the study was to develop teachers' understanding of children's mathematical thinking and to see if and how they used this knowledge to support all learners. Recognising children's mathematical learning as a sense-making process within the context of the classroom, and at the same time recognising that in developing a clear picture of the understanding of individual children teachers may need to think about how to best support particular children, reflects the reality of classroom practice and the challenge with which many teachers are presented. There was evidence, following professional development in CGI, of growth in teachers' knowledge about the mathematical understanding of individual children. This is consistent with earlier studies in children's mathematics (Boaler, 2002; Empson & Junk, 2004; Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001; Lampert, 2001; Vacc & Bright, 1999). This growth in knowledge of children's mathematical thinking supported the participating teachers to develop more inclusive pedagogical approaches to mathematics teaching. The important relationship between learning in meaningful contexts and the development of more equitable classrooms is consistent with international policies and practices promoting inclusive education (Allan, 2010; Armstrong et al., 2010; Slee, 2013; Thomas, 2013). Similarly in elementary mathematics classrooms in which lessons are structured to include all children purposefully, there are clear connections between teaching content, teachers' pedagogical knowledge and knowledge of children's understanding (Boaler, 2008; Lubienski, 2000).

At the outset of the study most of the teachers felt ill-equipped to support children who struggled in their mathematical learning. Teachers' anxiety about dealing with diversity in the classroom is well-recognised in research (Avramidis & Norwich, 2002). Yet it was notable that while developing problem-based contexts for learning through CGI no teacher suggested that particular children should be removed or work in isolation in ways that were described in the initial interviews. Instead they concerned themselves with their own practice and the content and structure of their teaching. Arguably this represents aspects of the 'shift in pedagogical thinking' called for by Florian and Black-Hawkins (2011). This is a significant shift, rooted in the experience of working with and watching children. The teachers were beginning to recognise the importance of their role in supporting all children in learning with understanding (Jordan et al., 2009). There was less evidence of them seeing the problem as a 'within-child deficit' and greater acknowledgement of their own role and responsibility. In fact some teachers were quite self-deprecating in this respect. It is important not to attribute blame and to recognise issues related to teacher professional learning in the area of children's mathematics as the development of the practice of teaching (Lampert, 2001; Stigler & Hiebert, 2009).

CGI provided a pedagogical framework underpinned by a constructivist philosophy which facilitated the development of teachers' pedagogical content knowledge by encouraging the teachers to learn not only about the pupils but also about themselves by reflecting on their existing beliefs and practices. In this way teacher learning was a socially participative process situated in the context of classroom experiences (Lave & Wenger, 1991); this reflected the findings of earlier studies (Carpenter et al., 1989; Franke & Kazemi, 2001).

Opportunities for teachers to collaborate and to reflect upon and analysis their own, and their peers', teaching and children's engagement (Azevedo, diSessa, & Sherin, 2012; Webb et al., 2014) is an important aspect of teacher learning situated in practice (Hiebert et al., 2002; Lewis et al., 2009) which promotes more inclusive classrooms (Sherin, Mendez, & Louis, 2004; Ylonen & Norwich, 2012).

Data from the final interviews show that all the participating teachers considered themselves to be more knowledgeable about children's mathematical thinking. Developing a clearer picture of children's conceptualisations supported a shift away from the transmission of knowledge and procedures and towards encouraging pupils to make connections in their mathematical thinking (Askew et al., 1997). Teachers' recognition of their knowledge of children's mathematical thinking was tempered by their concerns about how to make use of this knowledge. The ability to use this knowledge is a crucial step which several teachers found challenging. This was characterised by the question 'where next?' It is not an easy step, noticing and responding to children's solution strategies is theoretically informed practice which requires time (Carpenter et al., 1999; Jacobs et al., 2010; Van Oers, 2013). Prior to the introduction of CGI this had not been an issue, next steps were identified through planning frameworks. The principle of using knowledge of children's thinking to inform teaching and of accessing children's conceptual understanding through purposeful interactions, reflects the pedagogical approach that Watson has

argued for as an effective way of supporting children with learning difficulties (Watson, 1996). Furthermore, within the context of a community of learners it supports an inquiring stance that is a hallmark of inclusive pedagogy (Ainscow et al., 2006; Booth & Ainscow, 2002).

The notion that the expertise to support particular learners is beyond the scope of the regular teacher is a disempowering one that constrains inclusive practice. This concern was reflected in the initial interviews.

At the outset of the study the participating teachers subscribed to a traditional approach to support for pupils who struggled to learn (Thomas & Loxley, 2007). This stance corresponds to a within-child deficit model in which support is seen as a mechanism, external to existing practice, that can be put in place and recognisable as something additional and different. Responding to difference in this 'diagnostic-prescriptive' manner (Florian & Rouse, 2009; Ysseldyke, 2001) lacks critical understanding of children's thinking and fails to recognise the capacity of this understanding to inform teaching and support learning (Carpenter et al., 1989). It is perhaps unsurprising that teachers respond in this manner. This has been the dominant orthodoxy in special education for over a century (Dyson, 2001; Thomas & Loxley, 2007; Tomlinson, 1988; Watson, 1996) with recommendations advocating practice situated within this paradigm (Carnine, 1997, 2000; Engelmann, 2005; Kirschner, Sweller, & Clark, 2006).

The existence of a knowledge-base unique to special education has been contested (Lewis & Norwich, 2005; Thomas & Loxley, 2007). In mathematics, a recent study of teachers in UK special schools found that they had only a fragmented knowledge of children's mathematical thinking and strategy use with subsequent educational responses based on intuition rather than on an informed knowledge base (Moscardini, 2013). It has been argued that, in mathematics teaching, there is a 'lack of an adequate pedagogy' that is responsive to learners and their needs (Ryan & Williams, 2007, p. 5). An alternative to relying on external 'expertise' involves the professional development of teachers in ways that allow them to respond dynamically to the needs of all learners. If we are to work towards McIntyre's concept of an inclusive pedagogy (2009) and schools which place no limit on a child's potential (Hart et al., 2004) then it is essential that teachers are aware of the capacity and responsibility that they have for supporting all learners (Florian & Rouse, 2009; Jordan et al., 2009).

Inclusive pedagogy then is not simply about what teachers do, nor is it a formulaic response to those children who are deemed to have been 'included'. Teachers make pedagogical moves on the basis of their knowledge and beliefs. Teacher engagement, in what has been defined as an inclusive pedagogy, is influenced by growth in their knowledge and consequent changes in beliefs. Leat and Higgins (2002) argue that for change to be effective it requires a 'practical and manageable step' (p. 72) that can be undertaken by teachers in the course of their work. They suggest that through, what they describe as 'powerful pedagogical strategies', teachers' beliefs can be positively affected.

The commitment of teachers in the present study and the extent to which they were learning in interaction was evident. The openness and non-prescriptive nature of CGI respected teachers' autonomy and the diversity of practice observed in the classrooms testified to this. As well as providing specific input on the principles of CGI, the development sessions, particularly later ones, provided some opportunity for teachers to reflect on their own practice and to discuss and share what was happening within their classrooms. Teachers reflecting through discussion, not only with their students but also with other teachers, is an important part of their learning. The practicalities of managing this in schools is challenging, yet it is essential if change in practice is to be meaningful and sustainable (Stigler & Hiebert, 2009). Given the participants' almost unanimous stated intention to continue developing CGI in their practice, a challenge for future and wider professional development, particularly in the UK, is to consider how classroom practice and teacher learning might be supported further so that it remains situated and is not perceived as a 'method' or a deliverable resource to be explicated through inservice training.

8. Conclusion

CGI provides a framework for conceptualising mathematics teaching and learning that facilitates a response to pupils' needs based on an informed knowledge of their conceptual understanding. This aspect of teacher learning is situated in classroom interactions. It contrasts with reductionist and remedial approaches to supporting learners that are based on a medical model of diagnosis and labelling with an ensuant response to the label (Ysseldyke, 2001). A pedagogy that sustains assessment that is dynamic and functional helps to illuminate learners' needs on the basis of actual current knowledge and understanding, rather than by deficit and the disparity between where the child is and where he or she might or should be within a particular curricular framework. This proposition acknowledges that for some children learning difficulties may be socially constructed, however it does not overlook the fact that for some children certain aspects of learning may be problematic. The root of these difficulties may be organic, cognitive, neurological, psychological or social. A pedagogical response does not seek to determine the root cause of these learning difficulties, rather it attempts to address them pragmatically through an interactionist process (Watson, 1996). Teacher and pupil learning are situated within this dynamic process.

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